V. Thom's transversality theorem

Recall, that we eventually want to prove Thm II.8.
For this we need a <u>strong</u> transversality theorem.

1. Prop: M, N mfs, $S \subset N$ closed submf. Then
$C^\infty_{\pitchfork S} = \{ f \in C^\infty(M,N) \mid f \pitchfork S \}$ is an open subset of $C^\infty(M,N)$ (in the Whitney C^1, and thus in C^∞-topology).

Proof:
Define $U \subset J^1(M,N)$ s.t. $C^\infty_{\pitchfork S} = M(U) = \{ f \mid j^1 f(M) \subset U \}$

Show that U is open.

Let $\sigma \in J^1(M,N)_{x,y}$ and $f: M \to N$ represent σ.

Then
$$\sigma \in U \iff y \notin S$$
$$\text{or } y \in S \text{ and } T_yN = T_yS + df_x(T_xM)$$

U is open because:
Let $U^c = J^1(M,N) \setminus U$ and $(\sigma_n)_{n \in \mathbb{N}}$ a sequence in U^c converging to some σ.

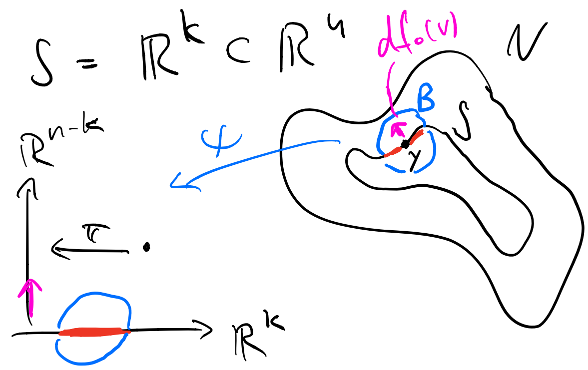
$x := s(\sigma)$, $y := t(\sigma)$. Since $t(\sigma_n) \in S \ \forall n \in \mathbb{N}$

and S closed, $y \in S$.

For f a representative of σ we choose coordinates (φ, A) around x and (ψ, B) around y s.t. $f(A) \subset B$ and $\psi(B \cap S) = \mathbb{R}^k \subset \mathbb{R}^n$.

So locally

$$M = \mathbb{R}^m, \quad x = 0, \quad N = \mathbb{R}^n, \quad S = \mathbb{R}^k \subset \mathbb{R}^n$$



For $\pi : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{R}^k = \mathbb{R}^{n-k}$

we have that

$$f \pitchfork S \text{ at } 0 \overset{!}{\iff} \pi \circ f \text{ is a submersion at } 0$$

$$\iff \pi \circ df_0 \notin \mathrm{Hom}^{<n-k}(\mathbb{R}^m, \mathbb{R}^{n-k})$$

Let $\Pi : \mathbb{R}^m \times S \times \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n) \underset{\text{closed}}{\subseteq} J^1(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^{n-k})$

$$(x, s, L) \mapsto \pi \circ L$$

$\mathrm{Hom}^{<n-k}(\mathbb{R}^m, \mathbb{R}^{n-k})$ is closed and Π is continuous, hence $\Pi^{-1}(\mathrm{Hom}^{<n-k}(\mathbb{R}^m, \mathbb{R}^{n-k}))$ is closed in $J^1(\mathbb{R}^m, \mathbb{R}^n)$.

But this is just U^c, because $\tau = (x, y, dg_x) \in U^c$

$$\overset{\text{loc.}}{\iff} y \in S \text{ and } g \not\pitchfork S \text{ at } 0$$

$\Leftrightarrow \quad \Pi(\tau) \in \mathrm{Hom}^{<n-k}(\mathbb{R}^m, \mathbb{R}^{n-k})$.

So, locally U^c closed, hence $\sigma = \lim_{n\to\infty} \sigma_n \in U^c$. $\blacksquare$

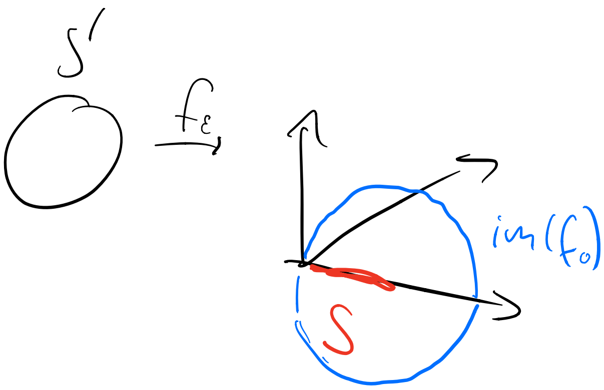
<u>Rem</u>: $S \subset N$ closed is really needed:

e.g. $M = S^1$, $N = \mathbb{R}^3$, $S = (0,1) \times \{(0,0)\} \subset \mathbb{R}^3$

$\rightsquigarrow \quad f \pitchfork S \iff \mathrm{im}(f) \cap S = \emptyset$

Take $f_\varepsilon : S^1 \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (x,y,0) + (1+\varepsilon, 0, 0)$



then $f_0 \pitchfork S$ but

$f_\varepsilon \not\pitchfork S \quad \forall \varepsilon > 0$!

<u>2. Lemma</u>:

M, N, T mfs, $S \subset N$ submf. Let $F : T \times M \to N$ smooth.

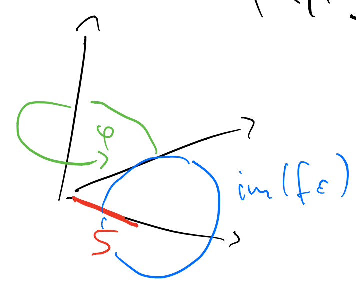
If $F \pitchfork S$, then the set $\{t \in T \mid F_t \pitchfork S\}$

is dense in T.

$F_t : M \to N \quad x \mapsto F(t,x)$

e.g.: $T = S^1$ $\quad F(\varphi, x, y) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} f_\varepsilon(x,y)$

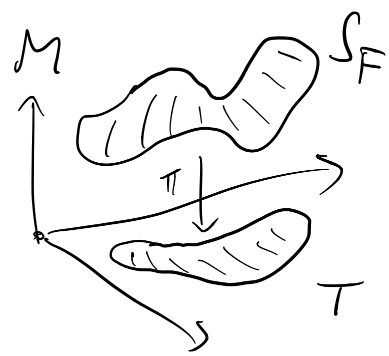
$F \pitchfork S$ and $F_\varphi \pitchfork S$ except for $\varphi = 0$.



Proof: $F \pitchfork S$, therefore

$S_F := F^{-1}(S)$ is a submf of $T \times M$ (Prop. II.11)

$\pi := \mathrm{proj}(T \times M \to T)|_{S_F} : S_F \to T$



1. If $t \notin \mathrm{im}(\pi)$, then

   $\mathrm{im}(F_t) \cap S = \emptyset$, so $F_t \pitchfork S$.

2. If $\dim S_F < \dim T$, then

   $\mathrm{im}(\pi)$ has measure zero in T

   and so on the dense set

$T \setminus \mathrm{im}(\pi)$ we have $F_t \pitchfork S$.

3. If $\dim S_F \geq \dim T$, then

<u>Claim:</u> t regular value of π

$\Rightarrow F_t \pitchfork S$.

Using <u>Sard's theorem</u>: X, Y smooth mfs.

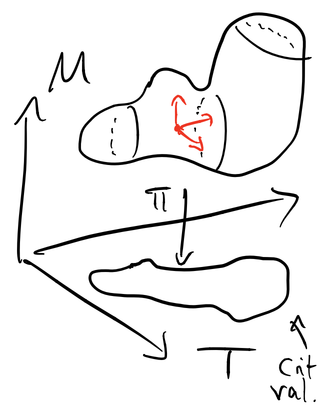
For $f: X \to Y$ smooth the set of critical values has measure zero in Y.

$\overset{\text{Corollary}}{\Rightarrow}$ set of regular values of f is is dense in Y.

this finishes the proof.

<u>Proof of claim:</u>

Let $t \in T$ be a regular value for π and $x \in M$.



If $(t, x) \in S_F$, then $F_t(x) \notin S$, so $F_t \pitchfork S$ at x.

Else, since t is regular and $\dim S_F \geq \dim T$ we have

$$T_{(t,x)}(T\times M) = T_{(t,x)}S_F + T_{(t,x)}(\{t\}\times M)$$

Apply $dF_{(t,x)}$ to get

$$\operatorname{im}(dF_{(t,x)}) = T_{F_t(x)}S + (dF_t)_x(T_xM)$$

By assumption $F \pitchfork S$ which means

$$T_{F(t,x)}N = T_{F(t,x)}S + dF_{(t,x)}(T_{(t,x)}(T\times M))$$

$$\Rightarrow T_{F_t(x)}N = T_{F_t(x)}S + (dF_t)_x(T_xM)$$

i.e. $F_t \pitchfork S$ at x.

$\blacksquare$

3. Corollary $\quad G: T\times M \to N$ smooth.

For $F(t,x) := j^kG_t(x)$ assume $F \pitchfork S$ where $S \subset J^k(M,N)$ subunf. Then $\{t\in T \mid j^kG_t \pitchfork S\}$ is dense in T.

4. <u>Thm (Thom):</u> M, N mfs, $S \subset J^k(M,N)$ submf. Then

$$J_S = \{ f \in C^\infty(M,N) \mid j^k f \pitchfork S \}$$ is a

residual subset of $C^\infty(M,N)$ in the Whitney

C^∞-topology.

<u>Proof:</u> To show: J_S is countable intersection of

open & dense subsets of $C^\infty(M,N)$.

1. Cover S by $(S_i)_{i \in \mathbb{N}}$ such that

   a) $\overline{S_i} \subset S$

   b) $\overline{S_i}$ is compact

   c) $\exists$ coordinate nbh. $U_i \subset M$, $V_i \subset N$ with
   $$(s \times t)(\overline{S_i}) \subset U_i \times V_i$$

   d) $\overline{U_i}$ compact

   (possible S is a submf and 2nd countable)

$\forall x \in M$ st $j^k f(x) \in \overline{S_i}$: $j^k f \pitchfork S$ at x

Define $J_{S_i} := \{ f \in C^\infty(M,N) \mid \boxed{j^k f \pitchfork S \text{ on } \overline{S_i}} \}$

Then $J_S = \bigcap_{i \in \mathbb{N}} J_{S_i}$

2. Each J_{S_i} open:

Define $\mathcal{J}_i := \{ g \in C^\infty(M, J^k(M,N)) \mid g \pitchfork S \text{ on } \overline{S_i} \}$

Same arguments as in proof of Prop. 1 show that $\mathcal{J}_i$ is open ($\overline{S_i}$ closed and contained in S)

Since $j^k : C^\infty(M,N) \to C^\infty(M, J^k(M,N))$ is continuous *, $J_{S_i} = (j^k)^{-1}(\mathcal{J}_i)$ is open.

3. Each J_{S_i} is dense:

Given the charts (from (c)) $\varphi : U_i \to \mathbb{R}^m$ and $\psi : V_i \to \mathbb{R}^n$ choose smooth functions

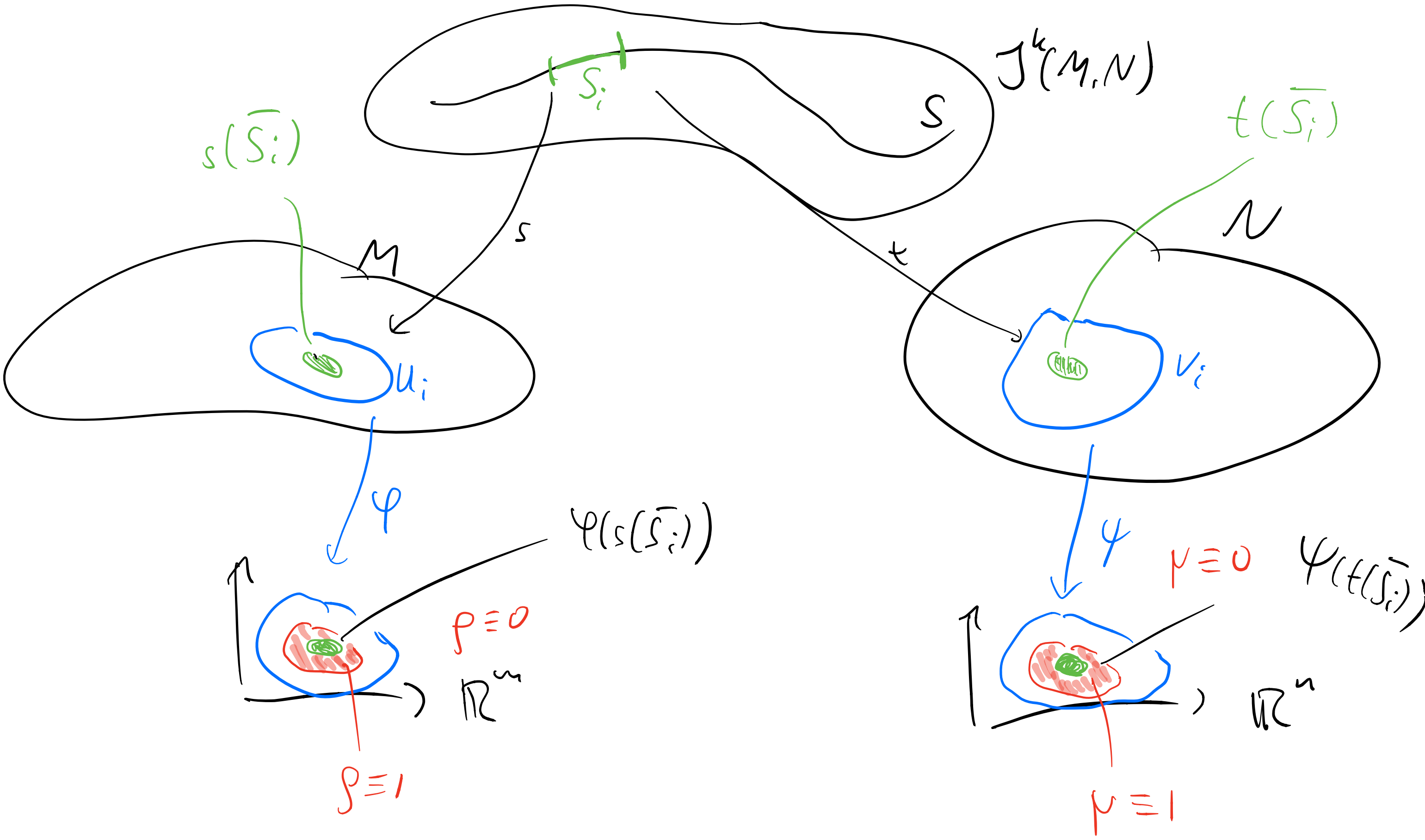
$\rho : \mathbb{R}^m \to [0,1] \subset \mathbb{R}$ and $\nu : \mathbb{R}^n \to [0,1] \subset \mathbb{R}$

with

$$\rho = \begin{cases} 1 & \text{in a nbh. of } \varphi(s(\bar{S}_i)) \\ 0 & \text{outside of } \varphi(U_i) \end{cases},$$

(possible bc. $\bar{S}_i$ compact)

$$\mu = \begin{cases} 1 & \text{in a nbh. of } \psi(t(\bar{S}_i)) \\ 0 & \text{outside of } \psi(V_i) \end{cases}$$



We want to use Corollary 3 to perturb any given $f \in C^\infty(M,N)$ to make it transversal to S on $\bar{S}_i$.

Let $P' = \{$ Polynomials $p: \mathbb{R}^m \to \mathbb{R}^n$ of degree $k \}$ and for $p \in P'$ define

$$G_p(x) = \begin{cases} f(x) & \text{if } x \notin U_i \text{ or } f(x) \notin V_i \\ \psi^{-1}\Big(\rho(\varphi(x)) \cdot \mu(\psi(f(x))) \cdot p(\varphi(x)) + \psi(f(x)) \Big) & \text{else} \end{cases}$$

This is a polynomial perturbation of f, outside domain of interest equal to f. It is smooth, as is $G: P' \times M \to N$ $(p,x) \mapsto G_p(x)$.

To apply Corollary 3 define $F(p,x) := j^k G_p(x)$ and show $F \pitchfork S$ on $\overline{S_i}$.

Claim: $\exists\, P \subset P'$ open nbh. of 0 so that $F: P \times M \to J^k(M,N)$ intersects S transversally on some nbh. of $\overline{S_i}$.

Proof: $**$.

If true, then given any $f \in C^\infty(M,N)$ there is a sequence $(p_\ell)_{\ell \in \mathbb{N}} \subset P$ converges to zero s.t.

$$j^k G_{p_\ell} \pitchfork S \quad \text{on} \quad \overline{S_i}$$

Since $G_0 = f$ and $G_{p_\ell} = f$ outside of $\overline{U_i}$, we conclude $\lim_{\ell \to \infty} G_{p_\ell} = f$ in the Whitney C^∞-topology, and therefore $\mathcal{J}_{S_i}$ is dense in $C^\infty(M,N)$. $\blacksquare$

Thom's transversality theorem asserts that a generic map is not only transverse to any given submf in the target mf but also to any condition put on its higher derivatives.

Moreover, perturbations to obtain transversality can be obtained in the space of k-jet extensions (and not in the space of all sections $M \to J^k$!).

<u>What does $j^k f \pitchfork S$ mean</u>? For instance, let $M = N = \mathbb{R}$, so $J^1(\mathbb{R},\mathbb{R}) \cong \mathbb{R}^3$, and $S = \mathbb{R}^2 \times \{0\}$.

$\sigma \in S \iff$ for $f: \mathbb{R} \to \mathbb{R}$ representing σ exists $x \in \mathbb{R}$ with $f'(x) = 0$

$j^1 f \pitchfork S$ at $x \iff T_{(x,f(x),0)} J^1(\mathbb{R},\mathbb{R}) = T_{(x,f(x),0)} S + d(j^1 f)_x (T_x \mathbb{R})$

$j^1 f(x) = (x, f(x), f'(x))$

$d(j^1 f)_x = (1, f'(x), f''(x))$

$\iff \mathbb{R}^3 = \langle (1,0,0), (0,1,0), (1,0,f''(x)) \rangle$

$\iff f''(x) \neq 0$ i.e. x <u>non-degen.</u> crit. pt.

<u>Holiday exercise</u>

- Find less trivial example
- Prove the following corollaries

<u>5. Corollary:</u>

For a family $\{S^i\}_{i \in \mathbb{N}}$ of submfs $S^i \subset J^{k_i}(M,N)$ the set $\{f \in C^\infty(M,N) \mid j^{k_i} f \pitchfork S^i\}$ is dense in $C^\infty(M,N)$. If the family is finite and each S^i closed, then this set is also open.

6. Corollary:

Let $S \subset J^k(M,N)$ subm. such that $\overline{s(S)}$ is contained in some $U \subsetneq M$ open. Let $f: M \to N$ smooth and V open nbh. of f in $C^\infty(M,N)$. Then there is $g \in V$ with $j^k g \pitchfork S$ and $g = f$ outside of U.

7. Corollary: Theorem II.12, the "weak" transversality theorem.